APPLICATIONS OF EXTREMAL THEOREMS IN GRAPH THEORY

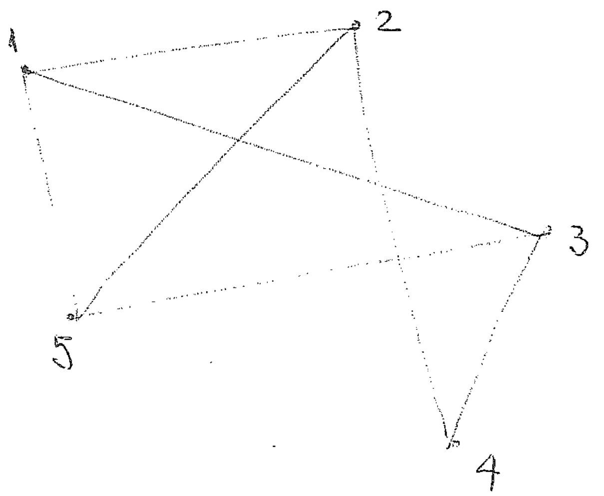
Defn. 1) A graph is a pair of sets $G=(V,E)$ where V is a finite set and $E \subseteq V\times V$. The elements of V are called vertices and the elements of E are called edges.

2) An edge $\{x,y\}$, $x,y\in V$ is usually written xy or yx.

3) Two vertices $x,y\in V$ are called adjacent or neighbours if xy is an edge of G.

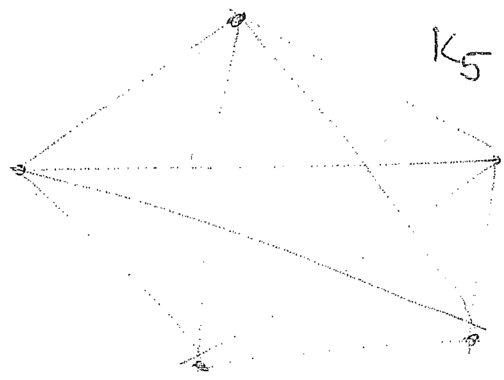
4) If all the vertices of G are adjacent then G is called complete. The complete graph with n vertices is denoted by K_n.

Defn Let $G=(V,E)$ and $G'=(V',E')$ be two graphs. We say that G' is a subgraph of G if $V'\subseteq V$ and $E'\subseteq E$.



$V=\{1,2,3,4,5\}$

$E=\{(1,2),(1,3),(1,5),(2,4),(2,5)(3,5)\}$



$G=(V,E)$

$G'=(V',E')$ subgraph of G

$V'=\{1,2,3,5\}$

$E'=\{(1,2),(1,5),(3,5)\}$.

Defn Let $G=(V,E)$ be a graph and $x\in V$. The degree of x, denoted by $d(x)$ is the nber of vertices of V which are adjacent with x

Remark: For any graph $G=(V,E)$ we have $\sum_{x\in V} d(x) = 2|E|$.

There are several people invited at a reunion. Show that the number of people having an odd number of friends is even.

Solution Consider the graph $G=(V,E)$ where V is the set of all people invited at the reunion and $xy\in E$ if x and y are friends. In this way $V=V_1\cup V_2$, where V_1, V_2 is the set of participants having an even, resp. an odd number of friends. Therefore $d(x)$ is even for any $x\in V_1$. Since

$$2|E| = \sum_{x\in V} d(x) = \sum_{x\in V_1} d(x) + \sum_{x\in V_2} d(x) = \text{even} + \sum_{x\in V_2} d(x)$$

we deduce that $\sum_{x\in V_2} d(x)$ is even. On the other hand $d(x)$ is odd for any $x\in V_2$ so $|V_2|$ must be even.

Theorems in Extremal Graph Theory

Thm 1 Let $G=(V,E)$ be a graph of order n with no complete subgraphs of order p. Then

$$\min_{x\in V} d(x) \le \left\lfloor \frac{p-2}{p-1} n \right\rfloor$$

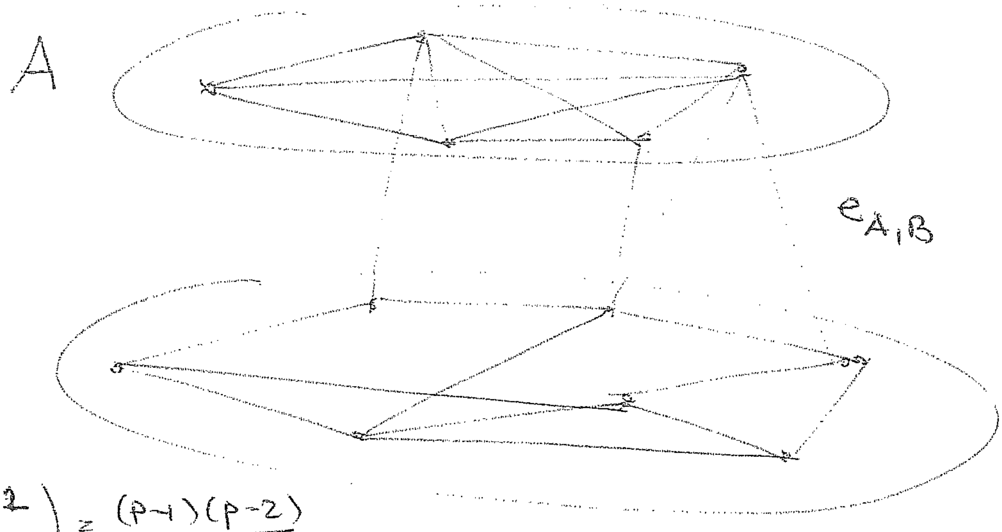
Thm 2 (TURÁN 1941) Let $G=(V,E)$ be a graph of order n with no complete subgraphs of order p. Then

$$|E| \le \frac{p-2}{p-1}\cdot\frac{n^2}{2}.$$

Proof. Induction over $n\geq 2$. If $n<p$ then it is easy to check that
$|E| \le \binom{n-1}{2} = \frac{(n-1)(n-2)}{2} \le \frac{p-2}{p-1}\cdot\frac{n^2}{2}$.

Let now $G=(V,E)$ be a graph of order $n\geq p$ having no subgraphs of K_p. By adding eventually edges, we may assumme that $G=(V,E)$ has the property $|E|$ is maximal. In particular, G contains a complete subgraph, say A, of order $p-1$.

Let $B=V\setminus A$ and denote by $e_A, e_B, e_{A,B}$ the numbers of edges of A, B, resp. connecting vertices from A to B



We have $e_A = \binom{p-1}{2} = \frac{(p-1)(p-2)}{2}$

By induction hypothesis, B is a graph of order $n-p+1$ with no complete subgraphs of order p. Hence $e_B \le \frac{p-2}{p-1} \cdot (n-p+1)^2/2$

Also any vertex from B is adjacent with at most $(p-2)$ vertices from A (otherwise we obtain a complete subgraph of G having order p).

Hence $e_{AB} \le (n-p+1)(p-2)$.

Therefore

$$|E| = e_A + e_B + e_{A,B} \le \frac{(p-1)(p-2)}{2} + \frac{p-2}{p-1}(n-p+1)^2/2 + (n-p+1)(p-2)$$

Problem 1. There are given 10 points in the plane which are vertices of a convex polygon. 26 of the lines that join these points are colored red. Show that there exists a triangle with vertices in these points and whose sides are al colored red.

Solution. Consider the graph $G=(V,E)$ where V is the set of all 10 points in the plane and we join two vertices x and y if the line xy was colored red. In this way $|V|=10$, $|E|=26$. We claim that G contains a complete subgraph of order 3. Assumming the contrary, from TURAN's Theorem we have

$$26 = |E| \le \frac{p-2}{p-1} \cdot \frac{n^2}{2} = \frac{1}{2} \cdot \frac{10^2}{2} = 25,$$ contradiction.

Problem 2. There are 19 teams in a competition. Each team played exactly 8 games. Show that at the end we can choose 3 teams such that none of them played against the other two teams.

Solution Consider the graph $G=(V,E)$ where V is the set of the 19 teams and $xy \in E$ if x didn't play against y. We must prove that G contains a complete subgraph K_3. Assuming the contrary, by Thm 1 we have

$$18-8=10=\min_{x\in V} d(x) \leq \frac{p-2}{p-1}\cdot n = \frac{1}{2}\cdot 19, \text{ contradiction.}$$

Problem 3. There are $2n$ mathematicians at a conference and each of them knows exactly k other participants. Find the minimum value of k such that one can always find 3 mathematicians that knows each other.

Solution. If $k\leq n$ the conclusion may not hold. We divide the participants into two sets of n mathematicians and we assume that any mathematician knows the participants in his group but no person from the other group.

For $k=n+1$ the conclusion holds by Thm 1 using the graph $G=(V,E)$ where V is the set of mathematicians and $xy\in E$ if x knows y

Problem 4. There are $2n$ points in the plane which are the vertices of a convex polygon. n^2+1 lines joining two of these points are colored red. Prove that we can find 4 points A, B, C, D such that at least five of the lines that pass through any two of them are colored red. (China Test, 1987)

Solution. Induction over $n \geq 2$.

For $n=2$ we have exactly the conclusion of the problem. Assume $n \geq 2$ and let $2(n+1)$ points such that $(n+1)^2+1 = n^2+2n+2$ lines determined by these points are colored red.

Consider the graph $G=(V,E)$ where V is the set of the $2(n+1)$ points and $xy \in E$ if the line joining x and y is red.

By Turan's Theorem we can find a complete subgraph $\{x,y,z\}$ of order 3. At least two of the vertices x,y,z have the degree of the same parity, say x and y.

If the set of the vertices $V \setminus \{x,y\}$ are joined by at lest n^2+1 then we can use the induction hypothesis in order to determine the four points A, B, C, D. If not, then the vertices of the set $V \setminus \{x,y\}$ are joined by at most n^2 edges which means that x and y are joined by at least $2n+2$ edges.

We claim that there exists another vertex $u \in V \setminus \{x,y,z\}$ that joins both x, y. Assuming the contrary, it follows that z is the only vertex of G that joins both x and y. Thus

$$d(x)+d(y)-1 \geq 2n+2 \Rightarrow d(x)+d(y) \geq 2n+3.$$

On the other hand, G has exactly $2n+2$ vertices, so

$$(d(x)-2)+(d(x)-2) \leq 2n-1 \Rightarrow d(x)+d(y) \leq 2n+3.$$

It follows that $d(x)+d(y) = 2n+3$ which contradicts the fact that $d(x)$ and $d(y)$ have the same parity.

Therefore, there exists $u \in V \setminus \{x,y,z\}$ that joins both x, y.

Hence x, y, z, u satisfy the requirement.

Problem 5 There are $3n$ distinct points in the plane such that the distance between any two of them is less than 1. Prove that at most $3n^2$ segments that join these points have the length greater than $1/\sqrt{2}$.

Solution We first claim that if A_1, A_2, A_3, A_4 are four distinct points such that $A_iA_j \le 1$, $1 \le i \le j \le 4$, then

$$\min_{1 \le i < j \le 4} A_iA_j < 1/\sqrt{2}.$$

Assuming the contrary we have to consider the following two cases:

I One of the points belongs to the interior of to the boundary of the triangle determined by the triangle determined by the other three.

Assume $A_4 \in [A_1A_2A_3]$. Then at least one of the angles $\widehat{A_1A_4A_2}$, $\widehat{A_2A_4A_3}$, $\widehat{A_3A_4A_1}$ has the measure $> \pi/2$. If for instanonce $\widehat{A_1A_4A_2} > \pi/2$ then by cosine theorem we have

$$A_1A_2^2 = A_1A_4^2 + A_2A_4^2 - 2\, A_1A_4 \cdot A_2A_4 \cdot \cos \widehat{A_1A_4A_2}$$
$$> A_1A_4^2 + A_2A_4^2 \ge 2 \cdot (1/\sqrt{2})^2 = 1, \text{ contradiction}$$

II A_1, A_2, A_3, A_4 are the vertices (not necessarily in this order) of a convex quadrilater. We proceed in the same manner since at least one of the angles of the convex quadrilater is $\ge \pi/2$

Consider now the graph $G=(V,E)$ where V is the set of all $3n$ points initially given and $xy \in E$ if the distance between x and y is $\ge 1/\sqrt{2}$.

According to the above arguments $\Longrightarrow$ G does not contain a complete subgraph of order 4. By Turan's Theorem it follows that

$$|E| \le \frac{2}{3} \cdot \frac{(3n)^2}{2} = 3n^2.$$

Problem 6 Given n distinct points on a circle of radius 1, prove that the number of line segments whose length is greater than $\sqrt{2}$ and whose endpoints lie in the given points on the circle is less than $n^2/3$

(Olimpiad Poland)

Solution. Use Turan's theorem as before by considering the graph $G=(V,E)$ where V is the set of all n points on the circle and E is the set of all segments joining them and whose lenght is $>\sqrt{2}$. Show first that G has no complete subgraphs of order 4.